

Calculus III: Coursework 4
M.A.H. MacCallum 2006

The deadline is 5PM on Tuesday Oct 24th. Hand-in is to the green box in the basement of the Maths building.

1. Sketch the curves whose parametric equations are

(a) $\mathbf{r} = (2 \cos \pi t)\mathbf{i} + (\sin \pi t)\mathbf{j}$

(b) $\mathbf{r} = (\sin \pi t)\mathbf{i}$

(c) $\mathbf{r} = t\mathbf{i} + |t|\mathbf{k}$

$(-\infty \leq t \leq \infty)$, and write down the derivatives $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$ where they are defined. [25]

Answer

(a) This is an ellipse with semi-major axes length 2 and 1 along the x and y axes respectively.

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= (-2\pi \sin \pi t)\mathbf{i} + (\pi \cos \pi t)\mathbf{j} \\ \frac{d^2\mathbf{r}}{dt^2} &= (-2\pi^2 \cos \pi t)\mathbf{i} + (-\pi^2 \sin \pi t)\mathbf{j}\end{aligned}$$

(Motion in each of x and y is simple harmonic.)

(b) The interval $[-1, 1]$ in x .

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \pi \cos \pi t \mathbf{i} \\ \frac{d^2\mathbf{r}}{dt^2} &= -\pi^2 \sin \pi t \mathbf{i}\end{aligned}$$

(Simple harmonic motion)

(c) The lines $z = -x$ for $x = t \leq 0$ and $z = x$ for $x = t > 0$ in the (x, z) plane.

In $t < 0$, $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{k}$ and in $t > 0$, $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{k}$. In both regions the second derivative is $\mathbf{0}$ (Note, the zero vector, not the scalar 0). The curve is not differentiable at $t = 0$.

Marks:

For curves 5,2,3 respectively. For evidence of understanding (in at least one of the parts) of how to form the derivatives 5, for correct answers for them 4,2,4 respectively. Lose 1 in the last part unless it is clear curve is not differentiable at $t = 0$.

2. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = 3y^2\mathbf{i} - 2x\mathbf{j} + 4xz\mathbf{k}$, over each of the following curves from $(0, 0, 0)$ to $(1, 1, 1)$:

(a) $C: \mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \quad 0 \leq t \leq 1$

(b) $C: \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k} \quad 0 \leq t \leq 1$

(c) C consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the line segment from $(1, 1, 0)$ to $(1, 1, 1)$.

[25]

Answer

(a) On this curve $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})dt$ and $\mathbf{F} = 3t^2\mathbf{i} - 2t\mathbf{j} + 4t^2\mathbf{k}$ so we have

$$\int_0^1 (3t^2 - 2t + 4t^2)dt = [7t^3/3 - t^2]_0^1 = 4/3.$$

(b) On this curve $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k})dt$ and $\mathbf{F} = 3t^4\mathbf{i} - 2t\mathbf{j} + 4t^5\mathbf{k}$ so we have

$$\int_0^1 (3t^4 - 4t^2 + 16t^8)dt = [16t^9/9 + 3t^5/5 - 4t^3/3]_0^1 = 47/45.$$

(c) First part is $\mathbf{r} = t\mathbf{i} + t\mathbf{j}$ and the second part is $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, each with $0 \leq t \leq 1$. On first part $\mathbf{F} = 3t^2\mathbf{i} - 2t\mathbf{j}$ and on the second part $\mathbf{F} = 3\mathbf{i} - 2\mathbf{j} + 4t\mathbf{k}$. So from the first part we get

$$\int_0^1 (3t^2 - 2t)dt = [t^3 - t^2]_0^1 = 0$$

while the second part gives

$$\int_0^1 4t dt = [2t^2]_0^1 = 2$$

which is also the total.

Marks: For evidence of correct method for forming $d\mathbf{r}$ 5, for correct method of evaluating $\mathbf{F}$ on a curve and forming the integral 5, for accuracy 5 per part.

Comment: In the next part of the course we discuss further the fact that the values of such integrals depend on the path taken, not just the endpoints.

3. A magnetic field exerts a force $\mathbf{F} = -\alpha(x\mathbf{i} + y\mathbf{j})$ (where α is a constant) on a charged particle, making the particle move along a spiral path

$$\mathbf{r} = (\cos \pi t)\mathbf{i} + (\sin \pi t)\mathbf{j} + t\mathbf{k} \quad (0 \leq t \leq 5).$$

Find the work $\int \mathbf{F} \cdot d\mathbf{r}$ done by the force.

[10]

Answer

On the curve $d\mathbf{r} = [(-\pi \sin \pi t)\mathbf{i} + (\pi \cos \pi t)\mathbf{j} + \mathbf{k}]dt$ and $\mathbf{F} = -\alpha((\cos \pi t)\mathbf{i} + (\sin \pi t)\mathbf{j})$ so the integral is

$$\int_0^5 \pi \alpha (\sin \pi t \cos \pi t - \cos \pi t \sin \pi t) dt = 0.$$

Marks: For correct method and forming integral 5, for accuracy 5.

4. If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, evaluate

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dS$$

where

- (a) S is the rectangular box formed by the planes

$$x = \pm a, \quad y = \pm b, \quad z = \pm c.$$

- (b) S is the sphere of radius a , centre the origin.

In both cases, the outward normal $\mathbf{n}$ should be taken.

[25]

Answer

- (a) On the surfaces $x = \pm a$ the normals are $\pm\mathbf{i}$ respectively, and $\mathbf{F} \cdot \mathbf{n}$ is $\pm x = a$ on both surfaces so the total for these two surfaces is

$$2 \int_{-b}^b \int_{-c}^c a \, dz \, dy = 8abc.$$

The other pairs of faces similarly give the same result twice more and so the total is $24abc$.

- (b) On the sphere $\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2} = \mathbf{e}_r = \mathbf{F}/a$ so we get

$$\int_S a \, dS = 4\pi a^3$$

(using the area of the sphere).

Marks: (a) correct methods of forming integrals needed 5, accuracy 6 (b) forming correct $\mathbf{n}$ and therefore integral 9, accuracy in evaluation 6.

5. If $f = xyz$, evaluate $\int_V f \, dV$ where

- (a) V is the rectangular volume bounded by the planes

$$x = 0, a, \quad y = 0, b, \quad z = 0, c,$$

- (b) V is the rectangular volume bounded by the planes

$$x = \pm a, \quad y = \pm b, \quad z = \pm c.$$

[15]

Answer

- (a)

$$\int_0^a \int_0^b \int_0^c xyz \, dz \, dy \, dx = a^2 b^2 c^2 / 8.$$

- (b)

$$\int_{-a}^a \int_{-b}^b \int_{-c}^c xyz \, dz \, dy \, dx = 0.$$

Marks: Method correct in at least one part 5, accuracy 5 per part.